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DYNAMICAL NONCOMMUTATIVITY AND NOETHER THEOREM IN TWISTED $\phi^{\star 4}$ THEORY

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Abstract

A $\star$-product is defined via a set of commuting vector fields $X_a = e_a{}^\mu(x)\partial_\mu$, and used in a $\phi^{\star 4}$ theory coupled to the $e_a{}^\mu(x)$ fields. The $\star$-product is dynamical, and the vacuum solution $\phi = 0$, $e_a{}^\mu = \delta_a^\mu$ reproduces the usual Moyal product. The action is invariant under rigid translations and Lorentz rotations, and the conserved energy-momentum and angular momentum tensors are explicitly derived.

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1 Introduction

Noncommutative coordinates are a recurrent theme in mathematical physics. Early considerations on quantum phase space geometry can be found in [1], and the idea of noncommuting spacetime coordinates goes back to Heisenberg who suggested (in a letter to Peierls [2]) that uncertainty relations between spacetime coordinates could resolve the UV divergences arising in quantum field theories. This motivation still holds today, in particular for nonrenormalizable field theories of gravity where finiteness is the only option for consistency.

The issue was explored initially by Snyder in [3], and since then noncommutative geometry has found applications in many branches of physics, in particular in the last two decades. Some comprehensive reviews can be found in references [4], [5], [6], [7], [8],[9], [10]. As an important example, the development of the noncommutative differential geometry on quantum groups (continuous deformations of Lie groups) and more generally on Hopf algebras, has led to interesting generalizations of gauge and gravity theories, whose symmetries are deformations of the corresponding classical symmetries (see for ex. [11],[12], [13],[14], [15],[16],[17]).

On the other hand, string theories have been pointing towards a non-commuting scenario already in the 1980's [18]. Later Yang-Mills theories on noncommutative spaces have emerged in the context of M-theory compactified on a torus with a constant background 3-form field, or as the low-energy limit of open strings in a background B-field describing the fluctuations of the D-brane worldvolume [19].

Noncommutative spacetime is described in terms of coordinates $\hat{x}^\mu$,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \tag{1.1}$$

where $\theta^{\mu\nu}$ is an antisymmetric tensor, usually chosen to be constant (corresponding to constant background fields in string theory).

The algebra of functions of these noncommuting coordinates can be represented by the algebra of functions on *ordinary* spacetime, equipped with a noncommutative $\star$-product. For constant $\theta^{\mu\nu}$ it is known as the Groenewold-Moyal product [20], [21]:

$$f(x) \star g(x) \equiv e^{\Delta}(f, g), \quad \Delta(f, g) \equiv \frac{i}{2}\theta^{\mu\nu}(\partial_\mu f)(\partial_\nu g). \tag{1.2}$$

and indeed reproduces (1.1) as $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$. This $\star$-product is associative and noncommutative, and was first introduced to represent on the classical phase space the product of quantum operators.

Field theories on noncommutative spacetime can then be obtained by replacing the usual product between fields with the $\star$-product. Because of the non-polynomial character of the $\star$-product the resulting field theories are non-local.[1] Thus deformations of scalar, Yang-Mills, gravity theories have been considered, and (at least

[1]The realization (1.2) of the $\star$-product $f \star g$ holds for a limited class of functions (e.g. polynomials, or analytic and rapidly decreasing functions). For a richer class of functions, e.g. smooth and rapidly decreasing (Schwarz test functions), an integral representation of the $\star$-product is needed. One such representation is $f \star g\,(x) = (2\pi)^{-2D} \iint f(x + \frac{1}{2}\theta u) g(x+s) e^{ius} \mathrm{d}^D u\, \mathrm{d}^D s$ [22] and explicitly encodes the nonlocality of the $\star$-product.

in the first two cases) their quantum behaviour is under active investigation, see for ex. [23, 24] and references therein.

The $\star$-deformation usually leads to $\star$-deformed invariances of the non-local actions: for example $U(n)$ $\star$-Yang-Mills theory is invariant under $\star$-gauge transformations on the fields $\delta_\epsilon A_\mu = \partial_\mu \epsilon - i(A_\mu \star \epsilon - \epsilon \star A_\mu)$. Spacetime symmetries are likewise deformed: while the non-local actions are invariant under rigid translations (so that a conserved energy momentum tensor can be found via the usual Noether theorem), Lorentz symmetry is typically broken, since the constant antisymmetric tensor $\theta^{\mu\nu}$ cannot be a Lorentz invariant tensor in $D > 2$, cf. [25],[26],[27],[28],[29], and [30]. However the $\star$-deformed action is invariant under a deformed Lorentz symmetry, acting on $\star$-products of fields with a deformed Leibniz rule [31, 29]. In this case the usual Noether theorem for global Lorentz rotations does not apply, and no conserved charge has been found so far.

In the present paper we propose a way to restore exact (undeformed) Lorentz symmetry in a $\star$-deformed interacting scalar theory. The key ingredient is a generalized Moyal product, defined via a set of commuting vector fields $X_a = e_a{}^\mu(x)\partial_\mu$ as given in eq. (2.2). This product corresponds to the twist $\mathcal{F} = \exp[-\frac{i}{2}\theta^{ab} X_a \otimes X_b]$. It gives rise to a twisted scalar field theory where $e_a{}^\mu$, and hence the $\star$-product itself, becomes dynamical. The condition $[X_a, X_b] = 0$ implies constraints on $e_a{}^\mu$, that can be solved off-shell in terms of D scalar fields ϕ^a. Thus the dynamical $\star$-product is well defined off-shell.[2] Field theories on noncommutative spaces (1.1) with nonconstant $\theta^{\mu\nu}(x)$ have been considered for example in ref.s [33, 34, 15, 24].

In Section 2 we determine the $e_a{}^\mu$ fields in terms of the ϕ^a, and specify the action of the $\phi^{\star 4}$ theory coupled to the ϕ^a fields. Section 3 contains the variations of the Lagrangian and the resulting field equations. In Section 4 Noether theorem is applied to derive conserved energy-momentum and angular momentum tensors. Section 5 contains some final considerations. Useful formulas are collected in the Appendix.

2 Dynamically twisted $\phi^{\star 4}$ theory

2.1 Generalized Moyal product

One of the most studied examples of noncommutative (deformed) spaces is the canonically deformed space or the θ-deformed space, see for ex. [9]. The deformation is contained in the associative and noncommutative Groenewold-Moyal $\star$-product given by

$$f \star g = \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g\}, \tag{2.1}$$

[2] A $\phi^{\star 4}$ action with spacetime dependent (but *nondynamical*) noncommutativity has been considered in [32].

where the map μ is the usual pointwise multiplication: $\mu(f \otimes g) = fg$. This product can be generalized as

$$\begin{aligned} f \star g &= \mu\{\mathcal{F}^{-1} f \otimes g\} \\ &= \mu\{e^{\frac{i}{2}\theta^{ab} X_a \otimes X_b} f \otimes g\} \\ &= e^{\Delta}(f, g), \end{aligned} \tag{2.2}$$

where θ^{ab} is a constant antisymmetric matrix, and the bilinear operator Δ is defined by (2.2) to act on a couple of functions as

$$\Delta(f, g) \equiv \frac{i}{2}\theta^{ab}(X_a f)(X_b g) \ . \tag{2.3}$$

(cf. Appendix). The twist $\mathcal{F}$ is given by

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{ab} X_a \otimes X_b} \tag{2.4}$$

and $X_a = e_a{}^\mu(x)\partial_\mu$ are D commuting vector fields, the index a being just a label for the vector fields. The coordinates x^μ span a D-dimensional Minkowski space with metric $\eta_{\mu\nu}$. In the commutative limit ($\theta^{ab} \to 0$) the product (2.2) reduces to the usual pointwise multiplication. The requirement that the vector fields X_a commute ensures the associativity of (2.2). From

$$[X_a, X_b] = 0 \tag{2.5}$$

we obtain the condition

$$e_{[a}{}^\nu \partial_\nu e_{b]}{}^\mu = 0. \tag{2.6}$$

Supposing that the square matrix $e_a{}^\mu$ has an inverse $e_\mu{}^a$ everywhere (so that the X_a are linearly independent), the condition becomes $\partial_{[\mu} e_{\nu]}{}^a = 0$ and it is solved by

$$e_\nu{}^a(x) = \partial_\nu \phi^a(x). \tag{2.7}$$

In this way the $\star$-product (2.2) is determined by D scalar fields subject to the condition that $\partial_\mu \phi^a$ is everywhere invertible. Since $X_a \phi^b = \delta_a^b$, the fields ϕ^b can be seen as new coordinates along the X_a directions. In particular, from (2.2) we find

$$[x^\mu \overset{\star}{,} x^\nu] \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{ab} e_a{}^\mu e_b{}^\nu. \tag{2.8}$$

Noncommutativity is here given by the space-time dependent (possibly degenerate) antisymmetric tensor

$$\Theta^{\mu\nu}(x) \equiv \theta^{ab} e_a{}^\mu(x) e_b{}^\nu(x) \ . \tag{2.9}$$

With this particular form of x-dependent noncommutativity parameter, originating from the twist (2.4), we have at our disposal the powerful twist machinery that allows to construct the differential calculus and geometry [15] relevant for the $\star$-product (2.2). The dimensionful parameters θ^{ab} can be considered fundamental constants (for example related to Planck length). Note also that $\Theta^{\mu\nu}$ transforms

as an antisymmetric tensor under usual Lorentz transformations $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, since ${e_a}^\mu(x)$ transforms as a vector. The relation $[x^\mu \stackrel{\star}{,} x^\nu] = i\Theta^{\mu\nu}(x)$ is therefore covariant under usual Lorentz transformations. The $\star$-product is invariant under Lorentz transformations, as it follows most easily from the Lorentz invariance of the vector fields $X_a = {e_a}^\mu(x)\partial_\mu$.

Due to (2.5) the action of X_a satisfies the Leibniz rule:

$$X_a(f \star g) = (X_a f) \star g + f \star (X_a g) \tag{2.10}$$

whereas a *deformed* Leibniz rule holds for the usual partial derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$ [15] .

2.2 Action

We use the $\star$-product (2.2) to define an action for $\phi^{\star 4}$ theory coupled to ϕ^c :

$$\begin{aligned} S[\phi, \phi^a] &= \int \Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi - \frac{m^2}{2}\phi \star \phi - \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \\ &\qquad + \frac{1}{2}\partial_\mu\phi_c \star \partial^\mu\phi^c\Big) \, \mathrm{d}^D x \ . \end{aligned} \tag{2.11}$$

Note however that the above integral is not cyclic: even with suitable boundary conditions at infinity

$$\int (f \star g) \ \mathrm{d}^D x \neq \int (g \star f) \ \mathrm{d}^D x \tag{2.12}$$

since $f \star g = g \star f + X_a(G^a)$ (see formula (6.5) in the Appendix where G^a is given explicitly) and $X_a(G^a)$ is *not* a total derivative. A cyclic integral can easily be defined by using the measure $e\,\mathrm{d}^D x$ where $e = \det({e_\mu}^a)$. Indeed $eX_a(G^a) = \partial_\mu(e{e_a}^\mu G^a)$ for any G^a, so that up to boundary terms:

$$\int (f \star g)\, e\, \mathrm{d}^D x = \int f g\, e\, \mathrm{d}^D x = \int (g \star f)\, e\, \mathrm{d}^D x \ . \tag{2.13}$$

The action (2.11) can then be rewritten by means of a cyclic integral:

$$\begin{aligned} S[\phi, \phi^a] &= \int \Big[\Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi - \frac{m^2}{2}\phi \star \phi - \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \\ &\qquad + \frac{1}{2}\partial_\mu\phi_c \star \partial^\mu\phi^c\Big) \star e^{-1}\Big] e\, \mathrm{d}^D x \ . \end{aligned} \tag{2.14}$$

Equation (2.13) allows to remove the $\star$-product in $\star\, e^{-1}$ and proves the equality of (2.11) and (2.14).

Unlike the ordinary Moyal case, the absence of the quartic potential $\phi^{\star 4}$ does *not* correspond to a free scalar theory, since only one $\star$-product can be removed in the remaining terms of (2.14). Then θ-dependent terms involve higher-order couplings between ϕ and ϕ^c fields.

World-index contractions being defined with the Minkowski metric $\eta^{\mu\nu}$, the action is invariant under global Lorentz transformations.

3 Variation of the Lagrangian and field equations

3.1 ϕ variation

We now derive the equations of motion for the fields ϕ and ϕ^a. To vary the action (2.14) with respect to the field ϕ we use the usual Leibniz rule, for example

$$\delta_\phi\Big(-\frac{m^2}{2}\int(\phi\star\phi\star e^{-1})e\,\mathrm{d}^Dx\Big)=-\frac{m^2}{2}\int((\delta\phi\star\phi+\phi\star\delta\phi)\star e^{-1})e\,\mathrm{d}^Dx. \quad (3.1)$$

The varied terms are grouped in the following way

$$\begin{aligned}\delta_\phi S &= \delta_\phi\int(\mathcal{L}_\star\star e^{-1})e\,\mathrm{d}^Dx\\ &= \int(\delta\phi\; E_\phi+\partial_\mu K^\mu)\;\mathrm{d}^Dx,\end{aligned} \quad (3.2)$$

with $\mathcal{L}_\star$ defined as

$$\mathcal{L}_\star=\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi-\frac{m^2}{2}\phi\star\phi-\frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi+\frac{1}{2}\partial_\mu\phi_c\star\partial^\mu\phi^c. \quad (3.3)$$

The equations of motion for the field ϕ are:

$$E_\phi=\frac{1}{2}\partial_\mu\Big(e\{\partial^\mu\phi\stackrel{\star}{,}e^{-1}\}\Big)+\frac{m^2}{2}e\{\phi\stackrel{\star}{,}e^{-1}\}+\frac{\lambda}{4!}e\{\phi\star\phi\stackrel{\star}{,}\{\phi\stackrel{\star}{,}e^{-1}\}\}=0. \quad (3.4)$$

In the commutative limit $\theta^{ab}\to 0$ this equation reduces to the usual field equation for the ϕ^4 theory

$$\Box\phi+m^2\phi+\frac{\lambda}{3!}\phi^3=0. \quad (3.5)$$

The current K^μ is given by

$$\begin{aligned}K^\mu \;=\;& \frac{e}{2}\delta\phi\{\partial^\mu\phi\stackrel{\star}{,}e^{-1}\}\\ &+ee_a{}^\mu\Big[T(\Delta)(\partial_\nu\delta\phi,\frac{1}{2}\tilde{X}^a\{\partial^\nu\phi\stackrel{\star}{,}e^{-1}\})\\ &\quad-\frac{m^2}{2}T(\Delta)(\delta\phi,\tilde{X}^a\{\phi\stackrel{\star}{,}e^{-1}\})\\ &\quad-\frac{\lambda}{4!}T(\Delta)\Big(\delta\phi,\tilde{X}^a(\{\phi^{\star 2}\stackrel{\star}{,}\{\phi\stackrel{\star}{,}e^{-1}\}\})\Big)\\ &\quad+S(\Delta)(\partial_\nu\phi,\tilde{X}^a(\partial^\nu\delta\phi\star e^{-1}))\\ &\quad-m^2S(\Delta)(\phi,\tilde{X}^a(\delta\phi\star e^{-1}))+\frac{\lambda}{12}S(\Delta)(\phi,\tilde{X}^a(\delta\phi\star\phi^{\star 2}\star e^{-1}))\\ &\quad-\frac{\lambda}{12}S(\Delta)(\phi\star\phi,\tilde{X}^a(\delta\phi\star\phi\star e^{-1}))\\ &\quad-\frac{\lambda}{12}S(\Delta)(\phi\star\phi\star\phi,\tilde{X}^a(\delta\phi\star e^{-1}))\Big]\end{aligned} \quad (3.6)$$

where $T(\Delta)$ and $S(\Delta)$ are operators defined in terms of Δ:

$$T(\Delta) \equiv \frac{\exp(\Delta) - 1}{\Delta}, \qquad S(\Delta) \equiv \frac{\sinh \Delta}{\Delta} \tag{3.7}$$

and $\tilde{X}^a \equiv \frac{i}{2}\theta^{ab}X_b$. Useful identities for the derivation of E_ϕ and K^μ are given in the Appendix.

3.2 ϕ^c variation

The variation of the action (2.14) with respect to the field ϕ^a has to be carried out carefully, since the field ϕ^a appears in the $\star$-product as well. One useful rule is (see the Appendix):

$$\delta_{\phi^c}(f \star g) = -(\delta\phi^c X_c f) \star g - f \star (\delta\phi^c X_c g) + \delta\phi^c X_c (f \star g) \tag{3.8}$$

where the functions f and g do not depend on ϕ^c. We group the terms in the following way

$$\begin{aligned} \delta_{\phi^c} S &= \delta_{\phi^c} \int (\mathcal{L}_\star \star e^{-1}) e \, \mathrm{d}^D x \\ &= \int \big(- \delta\phi^c (X_c\phi) E_\phi + \delta\phi^c E_{\phi^c} + \partial_\mu J^\mu \big) \, \mathrm{d}^D x. \end{aligned} \tag{3.9}$$

Therefore the equations of motion for the field ϕ^c read:

$$-(X_c\phi)\; E_\phi + E_{\phi^c} = 0 \tag{3.10}$$

with

$$\begin{aligned} E_{\phi^c} &= -\frac{1}{2}\partial_\mu(e\{\partial^\mu\phi_c \stackrel{\star}{,} e^{-1}\}) + \frac{1}{2}(X_c\phi)\partial_\mu(e\{\partial^\mu\phi \stackrel{\star}{,} e^{-1}\}) \\ &\quad + \frac{e}{2}(X_c\partial_\mu\phi^a)\{\partial^\mu\phi^a \stackrel{\star}{,} e^{-1}\} + \frac{e}{2}(X_c\partial_\mu\phi)\{\partial^\mu\phi \stackrel{\star}{,} e^{-1}\} - X_c\mathcal{L}_\star. \end{aligned} \tag{3.11}$$

When ϕ is on shell (i.e. $E_\phi = 0$), the ϕ^c field equations become simply

$$E_{\phi^c} = 0 \tag{3.12}$$

and reduce in the commutative limit to

$$\Box\phi^c = 0. \tag{3.13}$$

Note that the field equations (3.4),(3.11) are satisfied by the vacuum solution $\phi = 0$, ${e_\mu}^a \equiv \partial_\mu\phi^a = \delta^a_\mu$ (corresponding to the usual Moyal product). The field ϕ acts as a source for the noncommutativity field ϕ^c.

The current J^μ reads

$$\begin{aligned} J^\mu \;=\;& K^\mu(\delta\phi \to -\delta\phi^c X_c\phi) \\ &+\frac{e}{2}\delta\phi_c\{\partial^\mu\phi^c \stackrel{\star}{,} e^{-1}\} + \frac{e}{2}\delta\phi^c(X_c\phi)\{\partial^\mu\phi \stackrel{\star}{,} e^{-1}\} \\ &+e e_a{}^\mu(\delta\phi^a(\mathcal{L}_\star \star e^{-1}) - \mathcal{L}_\star \star (\delta\phi^a e^{-1})) \\ &+e e_a{}^\mu\Big[T(\Delta)(X_c\mathcal{L}_\star, \tilde{X}^a(\delta\phi^c e^{-1})) \\ &\qquad +T(\Delta)\Big(\partial_\sigma(\delta\phi^c e_c{}^\rho)\partial_\rho\phi, \frac{1}{2}\tilde{X}^a(\{\partial^\sigma\phi \stackrel{\star}{,} e^{-1}\})\Big) \\ &\qquad +T(\Delta)\Big(\partial_\sigma(\delta\phi^c e_c{}^\rho)\partial_\rho\phi_d, \frac{1}{2}\tilde{X}^a(\{\partial^\sigma\phi^d \stackrel{\star}{,} e^{-1}\})\Big) \\ &\qquad +S(\Delta)\Big(\partial_\sigma\phi, \tilde{X}^a((\partial^\sigma(\delta\phi^c e_c{}^\rho)\partial_\rho\phi)\star e^{-1})\Big) \\ &\qquad +S(\Delta)\Big(\partial_\sigma\phi_d, \tilde{X}^a((\partial^\sigma(\delta\phi^c e_c{}^\rho)\partial_\rho\phi^d)\star e^{-1})\Big)\Big]. \end{aligned} \tag{3.14}$$

4 Symmetries and conserved currents

Under a functional variation of the fields and a coordinate change:

$$\phi'(x) = \phi(x) + \delta\phi(x) \tag{4.1}$$
$$\phi'^c(x) = \phi^c(x) + \delta\phi^c(x) \tag{4.2}$$
$$x'^\mu = x^\mu + \epsilon^\mu \tag{4.3}$$

the variation of the action, to first order in $\delta\phi(x)$, $\delta\phi^c(x)$ and $\epsilon^\mu(x)$, is:

$$\delta S = \int \Big(\delta_\phi[(\mathcal{L}_\star \star e^{-1})e] + \delta_{\phi^c}[(\mathcal{L}_\star \star e^{-1})e] + \epsilon^\mu\partial_\mu[(\mathcal{L}_\star \star e^{-1})e] + (\mathcal{L}_\star \star e^{-1})e\;\partial_\mu\epsilon^\mu\Big)\,\mathrm{d}^D x \tag{4.4}$$

where we have used $\mathrm{d}^D x' = [1 + \partial_\mu\epsilon^\mu + O(\epsilon^2)]\mathrm{d}^D x$.

On shell, and integrated on an arbitrary manifold M (so that the total derivative terms do not disappear), this variation takes the form:

$$\delta S = \int_M \partial_\mu[K^\mu + J^\mu + \epsilon^\mu(\mathcal{L}_\star \star e^{-1})e]\;\mathrm{d}^D x. \tag{4.5}$$

4.1 Energy-momentum tensor

The action (2.14) is invariant under global translations, i.e. under the transformations:

$$\delta\phi = -\epsilon^\nu\partial_\nu\phi, \quad \delta\phi^c = -\epsilon^\nu\partial_\nu\phi^c, \quad \epsilon^\nu = \textit{constant}\,. \tag{4.6}$$

Substituting these variations into (4.5) leads to

$$0 = \delta S = \int_M \epsilon^\nu\partial_\mu T^\mu{}_\nu\;\mathrm{d}^D x. \tag{4.7}$$

where

$$
\begin{aligned}
T^{\mu}_{\ \nu} &= -\frac{e}{2}(\partial_\nu\phi)\{\partial^\mu\phi \stackrel{\star}{,} e^{-1}\} - \frac{e}{2}(\partial_\nu\phi_c)\{\partial^\mu\phi^c \stackrel{\star}{,} e^{-1}\} \\
&\quad + e e_a^{\ \mu}\Big(\mathcal{L}_\star \star (e^{-1}\partial_\nu\phi^a) - T(\Delta)(X_c\mathcal{L}_\star, \tilde{X}^a(e^{-1}\partial_\nu\phi^c))\Big) \qquad (4.8)
\end{aligned}
$$

is the conserved energy-momentum tensor. This tensor is not symmetric: only in the commutative limit it reduces to the canonical (and symmetric) energy-momentum tensor of the decoupled ϕ and ϕ^c fields.

From (4.8) the divergence $\partial_\mu T^{\mu}_{\ \nu}$ can be explicitly computed and shown to vanish on shell.

4.2 Angular momentum tensor

The action (2.14) is invariant under global Lorentz rotations, i.e. under the transformations:

$$
\delta\phi = -\epsilon^\nu\partial_\nu\phi = -\epsilon^{\nu\rho}x_\rho\partial_\nu\phi, \qquad \delta\phi^c = -\epsilon^\nu\partial_\nu\phi^c = -\epsilon^{\nu\rho}x_\rho\partial_\nu\phi^c, \qquad \epsilon^\nu = \epsilon^{\nu\rho}x_\rho \qquad (4.9)
$$

with $\epsilon^{\nu\rho}$ infinitesimal constant Lorentz parameter. Substituting into (4.5) yields

$$
0 = \delta S = \int_M \epsilon^{\nu\rho}\partial_\mu M^{\mu}_{\ \nu\rho}\, \mathrm{d}^D x. \qquad (4.10)
$$

where

$$
\begin{aligned}
M^{\mu}_{\ \nu\rho} &= \frac{e}{2}x_{[\nu}\partial_{\rho]}\phi\{\partial^\mu\phi \stackrel{\star}{,} e^{-1}\} + \frac{e}{2}x_{[\nu}\partial_{\rho]}\phi_c\{\partial^\mu\phi^c \stackrel{\star}{,} e^{-1}\} \\
&\quad - e e_a^{\ \mu}\Big(\mathcal{L}_\star \star (e^{-1}x_{[\nu}\partial_{\rho]}\phi^a)\Big) \\
&\quad + e e_a^{\ \mu}\Big[T(\Delta)(X_c\mathcal{L}_\star, \tilde{X}^a(e^{-1}x_{[\nu}\partial_{\rho]}\phi^c)) \\
&\qquad - T(\Delta)\Big(\partial_{[\nu}\phi, \frac{1}{2}\tilde{X}^a(\{\partial_{\rho]}\phi \stackrel{\star}{,} e^{-1}\})\Big) \\
&\qquad - T(\Delta)\Big(\partial_{[\nu}\phi^d, \frac{1}{2}\tilde{X}^a(\{\partial_{\rho]}\phi_d \star e^{-1}\})\Big) \\
&\qquad + S(\Delta)(\partial_{[\nu}\phi, \tilde{X}^a(\partial_{\rho]}\phi \star e^{-1})) \\
&\qquad + S(\Delta)(\partial_{[\nu}\phi_d, \tilde{X}^a(\partial_{\rho]}\phi^d \star e^{-1})). \qquad (4.11)
\end{aligned}
$$

is the conserved angular momentum tensor. In the commutative limit it reduces to the canonical angular momentum tensor of the decoupled ϕ and ϕ^c fields.

Again $\partial_\mu M^{\mu}_{\ \nu\rho}$ can be explicitly verified to vanish on shell.

5 Conclusions

By means of an extension of the Moyal product, we have implemented dynamical noncommutativity in $\phi^{\star 4}$ theory (in fact all the results hold also for $\phi^{\star n}$), and

simultaneously restored global Lorentz symmetry. This we achieved by introducing x-dependence in the definition of the $\star$-product in a factorized way, cf. (2.9), (2.2).

We have seen that the field equations of the resulting twisted ϕ^{*4} theory admit the vacuum solution $\phi = 0$, ${e_\mu}^a \equiv \partial_\mu \phi^a = \delta^a_\mu$. In particular when ${e_\mu}^a = \delta^a_\mu$ the $\star$-product between any two functions reduces to the Moyal product.

We conclude with the following observations:

1) The $\star$-product $f \star g$ differs from the ordinary product fg by terms involving partial derivatives of f, g and $\Theta^{\mu\nu}$ (containing ${e_a}^\mu$): then for slowly varying functions f, g and ${e_a}^\mu$ the $\star$-product is well approximated by the ordinary product. To be more precise, if the functions involved are approximately constant in cells of linear dimension $\sqrt{\theta}$ where θ is the order of magnitude of the θ^{ab} dimensionful constant parameters, and if ${e_a}^\mu$ is of the order of unity, then the $\star$-product can be replaced by the ordinary (commutative) product. Indeed in this case the typical terms in $f \star g - fg$ satisfy

$$\theta^{ab}({e_a}^\mu \partial_\mu f)({e_b}^\nu \partial_\nu g) \approx (\sqrt{\theta}\partial f)(\sqrt{\theta}\partial g) \approx 0 \tag{5.1}$$

2) There are other ways to deform $\phi^{\star 4}$ theory: indeed a kinetic term for the ϕ field

$$\int [{e_\mu}^a \star X_a(\phi) \star {e_\nu}^b \star X_b(\phi) \star e^{-1}]\eta^{\mu\nu} e \, \mathrm{d}^D x \tag{5.2}$$

still reduces to the usual kinetic term in the commutative limit. Note that ${e_\mu}^a \star X_a(\phi)$ is just the $\star$-Lie derivative along the vector field ∂_μ (see [15], Section 4). If we use this kinetic term in $\mathcal{L}_\star$, the resulting field equation for ϕ becomes:

$$X_a(\eta^{\mu\nu}\{{e_\mu}^b \star X_b(\phi) \mathbin{\overset{\star}{,}} e^{-1}\} \star {e_\nu}^a) + \frac{m^2}{2}\{\phi \mathbin{\overset{\star}{,}} e^{-1}\} + \frac{\lambda}{4!}\{\phi \star \phi \mathbin{\overset{\star}{,}} \{\phi \mathbin{\overset{\star}{,}} e^{-1}\}\} = 0. \tag{5.3}$$

In this equation all products are $\star$-products. We have not found a similar improvement for ϕ^c, essentially because the rule (3.8) involves ordinary products of the type $\delta\phi^c X_c f$. However ordinary products in the field equation (3.11) can be transformed into $\star$-products via the twist $\mathcal{F}$. Indeed, if we define

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b} \equiv \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha \tag{5.4}$$

$$\mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{ab}X_a \otimes X_b} \equiv \overline{\mathrm{f}}^\alpha \otimes \overline{\mathrm{f}}_\alpha \tag{5.5}$$

where $\mathrm{f}^\alpha, \mathrm{f}_\alpha, \overline{\mathrm{f}}^\alpha, \overline{\mathrm{f}}_\alpha$ are elements of the universal enveloping algebra of the X_a, then

$$g \star h = \overline{\mathrm{f}}^\alpha(g)\overline{\mathrm{f}}_\alpha(h) \tag{5.6}$$

so that

$$gh = \mathrm{f}^\alpha(g) \star \mathrm{f}_\alpha(h) \tag{5.7}$$

3) The extension of our results to include (noncommutative) gravity is under study. In this perspective we notice that the vector fields $X_a = e_a{}^\mu \partial_\mu$ are invariant not only under global Lorentz rotations, but also under general coordinate transformations.

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6 Appendix

We collect here some formulas used to derive the results of Sections 3 and 4.

Star product

$$
\begin{aligned}
f \star g &\equiv fg + \frac{i}{2}\theta^{ab}(X_a f)(X_b g) + \frac{1}{2!}\left(\frac{i}{2}\right)^2 \theta^{a_1 b_1}\theta^{a_2 b_2}(X_{a_1}X_{a_2} f)(X_{b_1}X_{b_2} g) + \cdots \\
&\equiv e^{\Delta}(f,g) \qquad (6.1)
\end{aligned}
$$

where powers of the bilinear operator Δ are defined as

$$
\Delta^n(f,g) \equiv \left(\frac{i}{2}\right)^n \theta^{a_1 b_1}\cdots\theta^{a_n b_n}(X_{a_1}\cdots X_{a_n} f)(X_{b_1}\cdots X_{b_n} g) \qquad (6.2)
$$

$$
(\Delta^0(f,g) \equiv fg) \qquad (6.3)
$$

From the definition (6.1) one finds the following identities (straightforward extensions of the identities derived in [27] for the usual Moyal product):

$$
\begin{aligned}
f \star g &= fg + X_a\Big[\frac{\exp(\Delta)-1}{\Delta}(f, \tilde{X}^a(g))\Big], && (6.4)\\
[f \overset{\star}{,} g] &\equiv f\star g - g \star f = 2X_a\Big[\frac{\sinh\Delta}{\Delta}(f, \tilde{X}^a g)\Big], && (6.5)\\
\{f \overset{\star}{,} g\} &\equiv f\star g + g\star f = 2fg + 2X_a\Big[\frac{\cosh\Delta - 1}{\Delta}(f, \tilde{X}^a g)\Big], && (6.6)
\end{aligned}
$$

with $\tilde{X}^a \equiv \frac{i}{2}\theta^{ab}X_b$.

Derivatives and variations

$$
\begin{aligned}
\delta_{\phi^c} e_a{}^\mu &= -e_a{}^\nu e_b{}^\mu \delta_{\phi^c} e_\nu{}^b = -e_a{}^\nu e_b{}^\mu \partial_\nu \delta\phi^b = -e_b{}^\mu X_a(\delta\phi^b), && (6.7)\\
\partial_\mu e &= e e_a{}^\nu \partial_\mu e_\nu{}^a = e e_a{}^\nu \partial_\nu \partial_\mu \phi^a = e X_a(\partial_\mu \phi^a), && (6.8)\\
\delta_{\phi^c} e &= e e_a{}^\nu \delta \partial_\nu \phi^a = e e_a{}^\nu \partial_\nu(\delta\phi^a) = e X_a(\delta\phi^a), && (6.9)\\
\delta_{\phi^c} X_a &= \delta_{\phi^c}(e_a{}^\mu \partial_\mu) = -e_b{}^\mu X_a(\delta\phi^b)\partial_\mu = -X_a(\delta\phi^b) X_b, && (6.10)\\
e X_a(f) &= \partial_\mu(e e_a{}^\mu f). && (6.11)
\end{aligned}
$$

In computing δ_{ϕ^c} variations, the following identity is useful:

$$\delta_{\phi^c}(f \star g) = -(\delta\phi^c X_c f) \star g - f \star (\delta\phi^c X_c g) + \delta\phi^c X_c(f \star g) \tag{6.12}$$

where the functions f and g do not depend on ϕ^c. This formula gives the $\delta\phi^c$ variation of a $\star$-product of two functions, due to the ϕ^c fields contained in the definition of $\star$, and can be proved by considering the variations of the typical term in $f \star g$:

$$\delta_{\phi^c}[(X_{a_1} \cdots X_{a_n} f)(\tilde{X}^{a_1} \cdots \tilde{X}^{a_n} g)] \tag{6.13}$$

By induction one proves easily that (3.8) holds for $\star$-products of an arbitrary number of factors:

$$\begin{aligned} \delta_{\phi^c}(f \star g \star \cdots \star h) \;\; = \;\; & -(\delta\phi^c X_c f) \star g \star \cdots \star h \\ & - f \star (\delta\phi^c X_c g) \star \cdots \star h - f \star g \star \cdots \star (\delta\phi^c X_c h) \\ & + \delta\phi^c X_c(f \star g \cdots \star h). \end{aligned} \tag{6.14}$$

Note: a different method to compute δ_{ϕ^c} variations is to recall that the ϕ^a fields determine the invertible transformation $x \to \varphi(x)$, i.e., in coordinates $x^\mu \to \phi^a(x)$. Any expression $V = V(\phi, X_a(\phi), X_a X_b(\phi), ...)|_x$ that depends on the scalar field ϕ and its X_a derivatives (like the potential term $\phi^{\star 4}$) can then be written as

$$V = V\left(\phi \circ \varphi^{-1}, (\phi \circ \varphi^{-1})_a, (\phi \circ \varphi^{-1})_{ab}, ...\right)|_{\varphi(x)} \tag{6.15}$$

where the indices $a, b, ...$ denote the partial derivatives $\frac{\partial}{\partial \phi^a}$, $\frac{\partial}{\partial \phi^b}, ...$. Under the variation $\phi^c \to \phi'^c = \phi^c + \delta\phi^c$ we have

$$\begin{aligned} V \;\; \to \;\; & V\left(\phi \circ \varphi'^{-1}, (\phi \circ \varphi'^{-1})_a, ...\right)|_{\varphi'(x)} \qquad (6.16) \\ & = V\left((\phi \circ \varphi'^{-1} \circ \varphi) \circ \varphi^{-1}, ((\phi \circ \varphi'^{-1} \circ \varphi) \circ \varphi^{-1})_a, ...\right)|_{\varphi(\varphi^{-1}(\varphi'(x)))} \; . \end{aligned}$$

In the last line the ϕ^c variation has been rewritten as a ϕ variation and a coordinate transformation. Infinitesimally these are given by

$$\begin{aligned} \delta_\phi^{(\delta\varphi)} \phi(x) \;\; & \equiv \;\; (\phi \circ \varphi'^{-1} \circ \varphi)(x) - \phi(x) = -\delta\phi^a X_a \phi(x) \; , \\ \delta_x^{(\delta\varphi)} x \;\; & \equiv \;\; \phi^{-1\mu}(\varphi'(x)) - x^\mu = \delta\phi^a X_a(x^\mu) \; . \end{aligned} \tag{6.17}$$

We conclude that the δ_{ϕ^c} variation equals the $\delta_\phi^{(\delta\varphi)}$ variation plus the variation generated by the vector field $\delta\phi^a X_a$,

$$\delta_{\phi^c} V = \delta_\phi^{(\delta\varphi)} V + \delta\phi^a X_a V \; . \tag{6.18}$$

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